Search-Inducing Informative Advertising

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Abstract

We study consumer search with informative advertising. In a duopoly with differentiated products, consumers are initially unaware of both the existence and the benefits of the products. A consumer that sees the ad of one brand will first explore that brand, but may then decide to search for an alternative. We find that if search costs are high, advertising mainly involves business stealing and firms overadvertise. If search costs are low advertising mainly involves market expansion and firms underadvertise as so firms tend to free ride. With more firms, our main results still hold.

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1 Introduction

Models of informative advertising typically assume that a consumer is looking for a particular product, and only knows that a firm sells that product when she sees it advertised by that firm. She then compares the firms from which she has seen an ad, and buys from the one offering the best deal (see in particular Butters, 1977, and Grossman and Shapiro, 1984). Yet, an ad may not just inform a consumer that a firm sells a particular brand: it may also inform her of the very existence and usefulness of that product.

Suppose for example that a consumer, while browsing the internet, sees an ad for a particular fitness tracker. Our consumer had never considered seeking out such a device, believing that her smartphone provides enough tracking capabilities. Yet, after seeing the ad, she checks out the tracker and decides that it may actually be a useful product. She is not particularly convinced by the brand she saw advertised though, and starts looking for alternatives. Eventually she settles for a product that competes with the one she originally saw advertised. In this case, the brand she eventually purchased has effectively free ridden on the advertising efforts of the brand that triggered her search.

It is exactly this scenario that we consider in this paper. We study a duopoly model of consumer search with differentiated products along the lines of Wolinsky (1986) or Anderson and Renault (1999). The costs of advertising are convex in the number of consumers a firm decides to inform. Consumers are initially unaware of either the existence or the benefits of the products. This changes if she sees them advertised. If she only sees an ad of one brand, she will check out that brand first but may then decide to investigate whether there is a second firm offering a similar product. If she receives ads from both firms, she will decide at random which firm to visit first.

Throughout this paper, we will use the above story as our leading example. But the insights of our paper easily generalize to any case where firms may free ride on the advertising effort of their competitor. An ad for a particular car, for example, may convince a consumer that it is time to replace her current vehicle—though she may ultimately decide to purchase a competitor's model instead. Seeing an ad for one brand might remind the consumer of advertisements from competing brands. In short, our model applies to any situation where informative advertising may induce consumers to search.

Traditionally, in this literature, "search costs" refer to the costs for a consumer of visiting a firm and checking out its product. In the context of our model, this is more subtle. There will be cases in which the consumer is informed about just one firm; it will then take her additional effort to find another firm offering a similar product. We thus assume that search costs are lower for a firm that is known to a consumer (in the sense that she has seen it advertised) rather than for one that is as of yet unknown to her.

A classic question in the literature on advertising is whether the market provides too much or too little advertising from a welfare perspective, see e.g. the survey in Bagwell (2007). We also ask that question in our model. The answer is not a priori clear. Firms may try to free ride on each other's advertising effort resulting in too little advertising. But the market may also provide a wasteful duplication of advertising effort leading to too much advertising.

Our framework allows us to cleanly distinguish between a market-expansion effect and a business-stealing effect of the advertising of an individual firm. If the costs of seeking out an unknown firm are high, firms have more market power vis-à-vis consumers that they inform. In that case, the market will provide overadvertising, as any additional ad mainly involves business stealing. But if search costs are low, there is likely to be underadvertising. Any additional ad now mainly involves market expansion, leading firms to try to free ride on each other's advertising efforts. In an extension, we show that this results also hold if there are more than two firms, provided we make some additional simplifying assumptions.

Our findings suggest that it may be welfare-improving to allow firms to coordinate their advertising efforts. For some parameter values, that is indeed the case. However, such a hypothetical cartel will always provide too little advertising. There are two reasons why that is the case. First, it cannot capture the full surplus generated by its advertising. Second, more advertising implies more competition and lower prices.

We are not the first to study the interplay of advertising and consumer search. Closely related to our paper is Haan and Moraga-González (2011), where the consumers' order of search is determined by the number of advertising messages they see from a particular firm. In that paper, advertising is purely combative; it does not affect the market outcome and hence there is always overadvertising. In our paper, advertising has a social function, as it informs consumers of the mere existence of a product.

Other work that looks at the relation between advertising and match values include Meurer and Stahl (1994), where firms can advertise match values, and Johnson and Myatt (2006), where advertising affects the distribution of match values. In Johnson (2009) firms can use ads to send signals about match values, but consumers can choose to block those ads. In Loginova (2009) consumers have an imperfect memory for the ads that they have seen. Papers that combine advertising and consumer search but in the context of homogeneous products include Robert and Stahl (1993), Janssen and Non (2008), McCarthy (2016), and Shelegia and Wilson (2021). Our work also builds on the literature on informative advertising, including Butters (1977) and, especially, Grossman and Shapiro (1984). That paper also studies a model in which each firm can choose to inform some fraction of consumers. We introduce search costs in that model, which allows us to effectively vary the extent of business-stealing.

The remainder of this paper is structured as follows. In Section 2, we present the model. We solve for the market equilibrium in Section 3. The problem of the social planner is considered in Section 4, where we also compare the market outcome with the social optimum. Section 5 illustrates our model for the case of a uniform distribution of match values and quadratic advertising costs. Section 6 studies the possibility of an advertising cartel, and compares that to both the market outcome and the social optimum. In Section 7, we study the case of an oligopoly with more than 2 firms. Section 8 concludes. All proofs are in the Appendix, unless noted otherwise.

2 The Model

In a nutshell, we study the following set-up. Consider a market where 2 firms sell differentiated products to a unit mass of consumers. Initially, consumers are uninformed about this market – or still have to be convinced about the merits of the product. Consumers face search costs. Firms can inform consumers by putting out ads. Doing so comes at a cost that is convex in the fraction of consumers that is informed. A consumer that does not see any ads will stay out of the market. A consumer that sees an ad for one firm will first check out that firm. If she likes the product but does not particularly care for the offering of that firm, she may seek out the other supplier. A consumer that sees ads of both firms will decide at random which firm to visit first. After having done so, she may still pay a visit to the other supplier. Crucially, we assume that the search costs of doing so are lower than in the case that she has not seen that firms' ads. After all, it is less costly to visit a firm that one has seen advertised rather than a firm that one still has to seek out.

On the consumer side, our model builds on the canonical model of search with differentiated products, see e.g. Wolinsky (1986) or Anderson and Renault (1999). A given consumer that buys from firm$j$ at price p_j, $j \in \{1, 2\}$, derives utility

$$u_j = v + \varepsilon_j - p_j. \tag{1}$$

The parameter v is the stand-alone utility of consuming the product: it is sufficiently high such that the market is always fully covered in equilibrium. This requires $v \geq 1$. In our welfare analysis, we will impose that $v = 1$: for very high v informing consumers would always be socially optimal, and we are interested in scenarios where that is not necessarily the case. The match value ε_j reflects how much this consumer likes the product of firm j, and is the privately observable realization of a random variable with distribution F and continuously differentiable density f with support normalized to $[0, 1]$. In line with the literature, we assume that $1 - F$ is log-concave. To learn the price p_j and their match value ε_j of firm j, a consumer must incur a search cost. The consumer searches sequentially with perfect recall.

Suppose that a consumer *only* receives an ad from firm k. She will then visit firm k first, and incurs search costs t to do so.[1] After having visited firm k, she may decide to search for an alternative firm that offers a similar product. We assume that the search costs of finding that firm and checking out its offer are s, and that this is known to consumers.[2] We assume that $s \geq t$: it is always (weakly) more costly for a consumer to investigate a product for which they have not seen an advertisement compared to one for which they have.

Suppose that a consumer receives ads from both firms. She will then pick one firm at random to visit first, and incurs search costs t to do so. After having visited that firm, the consumer may decide to also visit the other firm. As she has seen its ad, the costs of doing so are also t. A consumer that receives ads from neither firm will not be active on this market. Receiving

[1] We use the term 'visiting' loosely; this may also involve an online search for the exact specifications and price of the product.

[2] If a consumer is not entirely sure whether and how easily such a firm can be found, we may interpret s as the expected search costs,

some ad is necessary to trigger search; a consumer that receives no ad does not learn her about the existence of the product and her interest in it, and hence will not search in the first place.

On the supply side, firms set prices and advertising levels. An ad informs about the existence of the product sold in this market, and about the fact that this particular firm sells that product. Crucially, we assume that firms do not advertise their price, and also cannot provide information that would allow consumers to learn their match value for the particular offering of that firm.[3] We could however allow for that by setting $t = 0$. A firm can inform a fraction φ of consumers at a cost $c(\varphi)$, with $c'(\varphi) > 0$ if $\varphi > 0$, $c'(0) = 0$, and $c''(\varphi) > 0$ for all φ. The timing is as follows. First, firms simultaneously decide on advertising and prices.[4] Second, consumers search sequentially and make their purchase.

Note that our model nests others in the literature. With $c(\varphi) = 0$ both firms will advertise to all consumers and we are in the Anderson and Renault (1999) model. With $s \to \infty$ and $t = 0$ consumers only buy from firms they have seen advertised, and we are in the Grossman and Shapiro (1984) model.[5]. With $s = t = 0$, a consumer that has learned the existence of the product learns the specifics of both suppliers at zero costs, bringing them in the Perloff and Salop (1985) model.

3 Equilibrium

In this section, we solve for the equilibrium of our model. More precisely, we look for an equilibrium price p^* and an equilibrium fraction of consumers φ^* that each firm informs. To do so, we proceed as follows. First, for given t and s, we derive the optimal behavior for consumers. Second, for given consumer behavior, we derive the equilibrium p^* and φ^*.

Consumers Suppose that a consumer visits firm j first, and finds match value ε_j. Her expected benefit from visiting the other firm is given by $\int_{\varepsilon_j}^{1}(\varepsilon - \varepsilon_j)f(\varepsilon)d\varepsilon$. Suppose that this

[3]To allow for price advertising, we could use a directed search model as the basis for our analysis, see e.g. Haan et al. (2018). But that would greatly complicate the analysis without adding additional insight.

[4]Alternatively, we could assume that firms set advertising levels and prices sequentially. This, however, would not change the analysis much. More on that below.

[5]The only difference being the structure of match values: in Grossman and Shapiro (1984), firms are placed on a Salop (1979) circle, in our case match valued are iid distributed.

consumer has costs x of searching the second firm. That search is worthwhile only if the benefit exceeds x. We denote the ε_j for which this holds as $\hat{\varepsilon}_x$. Hence $\hat{\varepsilon}_x$ is implicitly defined by

$$x = \int_{\hat{\varepsilon}_x}^{1} (\varepsilon - \hat{\varepsilon}_x) f(\varepsilon) d\varepsilon. \tag{2}$$

With the usual arguments, such an $\hat{\varepsilon}_x$ always exists, provided that $x < E(\varepsilon_j)$.

Optimal prices in the benchmark model As a benchmark and to fix thoughts, consider first a model of search with differentiated products where all consumers have the same search costs x for both firms. This is the standard Anderson and Renault (1999) model. To avoid confusion, we denote demands and equilibrium prices in this model by $\bar{D}_{xj}$ and $\bar{p}_x^*$ respectively. Denoting $\Delta \equiv p_j - \bar{p}_x^*$, demand for firm j if it sets price p_j while the other firm sets the equilibrium price $\bar{p}_x^*$ is given by

$$\bar{D}_{xj}(p_j; \bar{p}_x^*) = \frac{1}{2} \left(1 - F(\hat{\varepsilon}_x + \Delta)\right) \left(1 + F\left(\hat{\varepsilon}_x\right)\right) + \int_{-\infty}^{\hat{\varepsilon}_x + \Delta} F\left(\varepsilon - \Delta\right) f\left(\varepsilon\right) d\varepsilon \tag{3}$$

This can be seen as follows. A consumer that visits firm j will buy there if $\varepsilon_{ij} > \hat{\varepsilon}_x + \Delta$. If j is visited first, it thus makes a sale with probability $1 - F(\hat{\varepsilon_x} + \Delta)$. If the other firm, k, is visited first, j will make a sale with probability $F(\hat{\varepsilon_x})(1 - F(\hat{\varepsilon_x} + \Delta))$, as this consumer first has to decline k's offer. Both firms are equally likely to be visited first, which yields the first term. The second term reflects consumers that find a match value that is too low at both firms, but return to j as they have the best match there.

Profits of firm j are $\pi_j \equiv p_j \cdot \bar{D}_{xj}$. Maximizing and imposing symmetry yields

$$\bar{p}_x^* \cdot \bar{D}'_{xj} + \bar{D}_{xj}\left(\bar{p}_x^*, \bar{p}_x^*\right) = 0,$$

with $\bar{D}'_{xj} \equiv \partial \bar{D}_{xj}\left(\bar{p}_x^*, \bar{p}_x^*\right)/\partial p_j$. Symmetry implies $\bar{D}_{xj}\left(\bar{p}_x^*, \bar{p}_x^*\right) = 1/2$, so $\bar{p}_x^* = -1/2\bar{D}'_{xj}$ which yields, using (3),

$$\bar{p}_x^* = \frac{1}{\left(1 - F(\hat{\varepsilon}_x)\right) f(\hat{\varepsilon}_x) + 2\int_0^{\hat{\varepsilon}_x} f^2(\varepsilon) d\varepsilon}. \tag{4}$$

Demand After ads are observed, we have four types of consumers. A fraction $\varphi_j \varphi_k$ observe ads from both firms, pick a firm at random to visit first, and face search costs t for each firm they visit. A fraction $\varphi_j(1 - \varphi_k)$ only observe an ad from firm j. They face search costs t when visiting j and an additional s if also visiting k. A fraction $(1-\varphi_j)\varphi_k$ only observe an ad from firm k and face search costs t when visiting k and an additional s if also visiting j. A fraction $(1-\varphi_j)(1-\varphi_k)$ observes no ads and does not buy the product. Total demand for firm j is

$$D_j(p_j, \varphi_j; p^*, \varphi^*) = \varphi_j \varphi_k \rho_{11} + \varphi_j(1-\varphi_k)\rho_{10} + (1-\varphi_j)\varphi_k \rho_{01}, \tag{5}$$

where ρ_{11} is the probability that a consumer that observed ads from both firms buys from j; ρ_{10} is the probability that a consumer that only observed ads from j buys from j, and ρ_{01} is the probability that a consumer that only observed ads from k buys from j.

Denoting $\Delta \equiv p_j - \bar{p}^*$, we have

$$\begin{aligned}
\rho_{11} &= \frac{1}{2}\left(1 - F(\hat{\varepsilon}_t + \Delta)\right)\left(1 + F\left(\hat{\varepsilon}_t\right)\right) + \int_{-\infty}^{\hat{\varepsilon}_t + \Delta} F\left(\varepsilon - \Delta\right) f\left(\varepsilon\right) d\varepsilon \\
\rho_{10} &= 1 - F(\hat{\varepsilon}_s + \Delta) + \int_{-\infty}^{\hat{\varepsilon}_s + \Delta} F\left(\varepsilon - \Delta\right) f\left(\varepsilon\right) d\varepsilon \\
\rho_{01} &= F\left(\hat{\varepsilon}_s\right)\left(1 - F(\hat{\varepsilon}_s + \Delta)\right) + \int_{-\infty}^{\hat{\varepsilon}_s + \Delta} F\left(\varepsilon - \Delta\right) f\left(\varepsilon\right) d\varepsilon
\end{aligned} \tag{6}$$

This can be seen as follows. The expression for ρ_{11} is identical to that in (3), with search costs $x = t$: both firms are equally likely to be visited first and search costs are t per firm. For ρ_{10}, note that these consumers visit firm j first and will buy there if $\varepsilon_{ij} > \hat{\varepsilon}_s + \Delta$, as continuing search entails an additional cost s. This yields the first term. The second term reflects consumers that find a match value that is too low at both firms, but return to j as they have the best match there. The expression for ρ_{01} is along the same lines. These consumers visit k first, and only visit j if the match value at k is too low—which happens with probability $F(\hat{\varepsilon}_s)$.

Equilibrium To find the equilibrium, suppose that firm k uses (p^*, φ^*), but j sets some (p_j, φ_j). Profits of firm j are

$$\pi_j(p_j, \varphi_j; p^*, \varphi^*) = p_j \cdot D_j(p_j, \varphi_j; p^*, \varphi^*) - c(\varphi_j). \tag{7}$$

Maximizing with respect to φ_i and p_j yields first-order conditions

$$\begin{aligned}\frac{\partial \pi_j}{\partial \varphi_j} &= p_j \frac{\partial D_j}{\partial \varphi_j} - c'(\varphi_j) = 0 \\ \frac{\partial \pi_j}{\partial p_j} &= D_j + p_j \frac{\partial D_j}{\partial p_j} = 0,\end{aligned}$$

Imposing symmetry and assuming an interior solution for φ^*, we have:

$$p^* \cdot \frac{\partial D_j(p^*, \varphi^*; p^*, \varphi^*)}{\partial \varphi_j} - c'(\varphi^*) = 0 \tag{8}$$

$$D_j + p^* \cdot \frac{\partial D_j(p^*, \varphi^*; p^*, \varphi^*)}{\partial p_j} = 0. \tag{9}$$

We first use (8) to derive the equilibrium value of φ. From (6), at equal prices we have[6]

$$\begin{aligned}\rho_{11} &= \frac{1}{2}, \\ \rho_{10} &= 1 - F(\hat{\varepsilon}_s) + \frac{1}{2} F(\hat{\varepsilon}_s)^2 \\ \rho_{01} &= F(\hat{\varepsilon}_s) \left(1 - \frac{1}{2} F(\hat{\varepsilon}_s)\right).\end{aligned}$$

Hence, from (3), after some further manipulations:

$$D_j(p^*, \varphi_j; p^*, \varphi_k) = \varphi_j - \frac{1}{2}\varphi_j\varphi_k - \frac{1}{2}(\varphi_j - \varphi_k) F(\hat{\varepsilon}_s)(2 - F(\hat{\varepsilon}_s)).$$

Taking the derivative with respect to φ_j and imposing symmetry we can write

$$\frac{\partial D_j(p^*, \varphi^*; p^*, \varphi^*)}{\partial \varphi_j} = \frac{1}{2}(1-\varphi) + \frac{1}{2}(1 - F(\hat{\varepsilon}_s))^2.$$

Condition (8) then yields an implicit expression for φ^*.

Next, we derive equilibrium prices using (9). From (5), at equal advertising levels we have, using (6),

$$\begin{aligned}D_j(p_j, \varphi^*; p^*, \varphi^*) &= \varphi^{*2}\left[\frac{1}{2}(1 - F(\hat{\varepsilon}_t + \Delta))(1 + F(\hat{\varepsilon}_t)) + \int_{-\infty}^{\hat{\varepsilon}_t + \Delta} F(\varepsilon - \Delta) f(\varepsilon)\, d\varepsilon\right] \\ &+ 2\varphi^*(1-\varphi^*)\left[\frac{1}{2}(1 - F(\hat{\varepsilon}_s + \Delta))(1 + F(\hat{\varepsilon}_s)) + \int_{-\infty}^{\hat{\varepsilon}_s + \Delta} F(\varepsilon - \Delta) f(\varepsilon)\, d\varepsilon\right]\end{aligned}$$

[6] Note that integration by parts implies $\int^{\hat{\varepsilon}_x} F(\varepsilon) f(\varepsilon) d\varepsilon = F(\hat{\varepsilon}_x)^2 - \int^{\hat{\varepsilon}_x} f(\varepsilon) F(\varepsilon) d\varepsilon$ hence $\int^{\hat{\varepsilon}_x} F(\varepsilon) f(\varepsilon) d\varepsilon = \frac{1}{2} F(\hat{\varepsilon}_x)^2$ which yields the expressions in the text.

which using (3) can be rewritten as:

$$D_j(p_j, \varphi^*; p^*, \varphi^*) = \varphi^{*2}\bar{D}_{tj} + 2\varphi^*(1-\varphi^*)\bar{D}_{sj}.$$

Note that we have $D_j(p^*, \varphi^*; p^*, \varphi^*) = \frac{1}{2}\varphi^{*2} + \varphi^*(1-\varphi^*)$: in a symmetric equilibrium, all consumers that are informed will buy, half of which will buy from from j. From (9),

$$p^* = \frac{\varphi^{*2} + 2\varphi^*(1-\varphi^*)}{2\left(\varphi^*\right)^2 \bar{D}'_{tj} + 4\varphi^*(1-\varphi^*)\bar{D}'_{sj}}. \tag{10}$$

This implies

Proposition 1 (Market equilibrium)**.** *Equilibrium advertising levels are given by*

$$c'\left(\varphi^*\right) = p^*\left[\frac{1}{2}\left(1-\varphi\right) + \frac{1}{2}\left(1 - F\left(\hat{\varepsilon}_s\right)\right)^2\right]. \tag{11}$$

The equilibrium price is the weighted harmonic mean of the equilibrium price when all consumers have search costs t (with weight φ^{*2}) and that when all consumers have search costs s (with weight $2\varphi^*(1-\varphi^*)$):

$$p^* = \frac{\varphi^{*2} + 2\varphi^*(1-\varphi^*)}{\frac{\varphi^{*2}}{\bar{p}_t^*} + \frac{2\varphi^*(1-\varphi^*)}{\bar{p}_s^*}} \tag{12}$$

Proof. Follows directly from the analysis above. ■

It is instructive to further analyze (11), the equilibrium condition on advertising. The left-hand side reflect the marginal cost of advertising. The right-hand side reflects the marginal benefit. This can be seen as follows. Suppose that a firm informs one additional consumer. With probability $1-\varphi^*$, this consumer was not yet informed by the other firm and hence will enter the market. This is a **market-expansion effect**. For now, assume that this new consumer is equally likely to buy from either firm. This yields the first term in square brackets.

Note However that a consumer is more likely to buy from the first firm that she visits. Denote this probability $\frac{1}{2}+\delta$, where δ is the **prominence bonus**. Suppose an ad reaches a consumer that was not yet informed. This yields an additional sale probability of δ on top of the $\frac{1}{2}$ we already considered. Now suppose an ad reaches a consumer that *was* already informed. Without this ad, this consumer would have bought from this firm with probability $\frac{1}{2}-\delta$. But when also seeing this ad, she is equally likely to end up buying from either firm. Hence, also then, the ad yields an additional sale probability of δ, as the competitor now loses their prominence bonus.

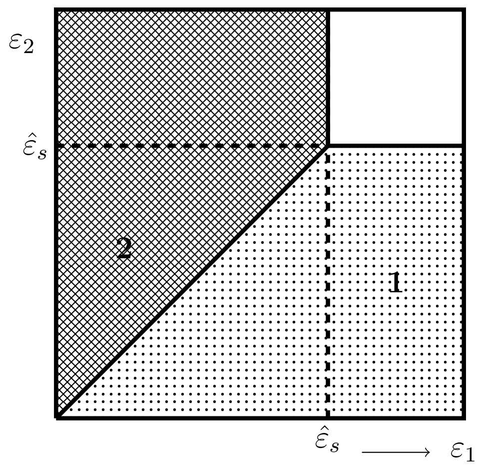


Figure 1: Deriving the prominence bonus

To pin down the prominence bonus δ, note the following. A consumer with $\varepsilon_1 < \hat{\varepsilon}_s$ and $\varepsilon_1 < \varepsilon_2$ will always end up buying from firm 2, regardless of which firm she visits first. This is the cross-hatched area in Figure 1. Similarly, a consumer with $\varepsilon_2 < \hat{\varepsilon}_s$ and $\varepsilon_2 < \varepsilon_1$ will always end up buying from firm 1. This is the dotted area in Figure 1. Only for consumers with $\varepsilon_1, \varepsilon_2 > \hat{\varepsilon}_s$ does the order of search matter: these consumers buy from the first firm they encounter. This applies to $(1-F(\hat{\varepsilon}_s))^2$ consumers – the white area in the north-east of Figure 1. Hence a firm that is visited first will sell to half of all consumers, plus an additional $\frac{1}{2}(1-F(\hat{\varepsilon}_s))^2$. This is our prominence bonus – and hence also the second term in square brackets in (11). This is a **business-stealing effect**.

From the Proposition we immediately have

Corollary 1. If $s > t$, equilibrium prices decrease as advertising levels increase. If $s = t$, equilibrium prices are independent of advertising levels.

Proof. Note that we can write

$$p^* = \bar{p}_t^* \cdot \bar{p}_s^* \cdot \frac{2-\varphi}{2\left(1-\varphi\right)\bar{p}_t^* + \varphi\bar{p}_s^*}.$$

This implies

$$\frac{\partial p^*}{\partial \varphi} = 2\bar{p}_t^* \cdot \bar{p}_s^* \cdot \frac{\bar{p}_t^* - \bar{p}_s^*}{\left(2\left(1-\varphi\right)\bar{p}_t^* + \varphi\bar{p}_s^*\right)^2}$$

With $\bar{p}_t^* < \bar{p}_s^*$, this expression is negative, while it equals zero if $\bar{p}_t^* = \bar{p}_s^*$. ■

Intuitively, an increase in advertising implies that more consumers will be informed by both firms. As a result, average search costs in the population will decrease, leading to lower prices.

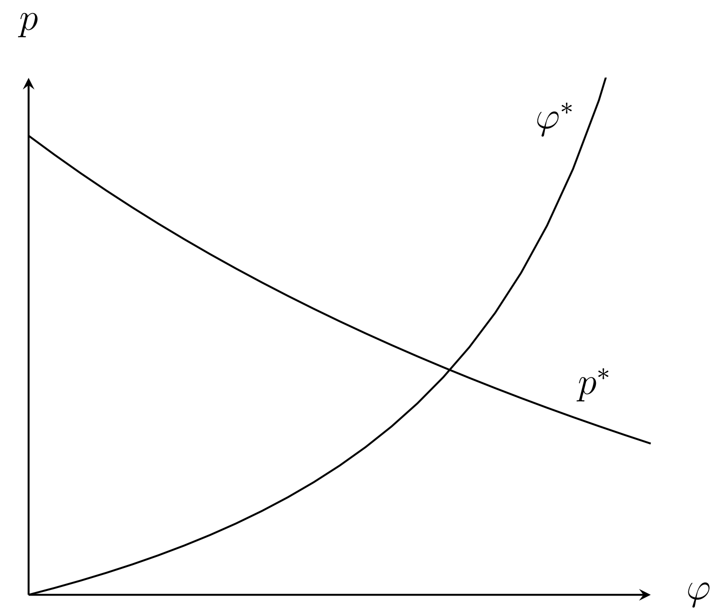


Figure 2: Finding the equilibrium.

The two curves determining the equilibrium are depicted in Figure 2.[7] The curve labelled φ^* reflects condition (11) in (φ, p)-space. It reflects that, ceteris paribus, higher prices will lead to higher equilibrium advertising levels: with higher prices, the returns to advertising will be higher, so firms will provide more of it. The curve labelled p^* plots condition (10). It reflects that, ceteris paribus, higher advertising levels imply lower prices – as we saw in Corollary 1.

Comparative statics are as follows:

Proposition 2 (Comparative statics market equilibrium)**.** *Equilibrium advertising φ^* is increasing in search costs s and t, and decreasing in the marginal costs of advertising. The equilibrium price p^* is increasing in t and in the marginal costs of advertising. The effect of an increase in s is ambiguous.*

These effects can easily be seen in Figure 2. An increase in s shifts the φ^*-curve to the right (higher search costs now induces firms to advertise more) and shifts the p^*-curve up (higher search costs imply higher prices). The total effect is for φ^* to increase while the effect on p^* is

[7] The Figure shows the case of a uniform distribution of preferences on $[0, 1]$, quadratic advertising costs $c(\varphi) = \varphi^2$, $t = 0.05$ and $s = 0.3$.

ambiguous. An increase in t only affects the p^*-curve, leading to an increase in both p^* and φ^*. An increase in the marginal costs of advertising shifts the φ-curve to the left, hence lowering the amount of advertising, but raising prices.

The effects of search costs on equilibrium advertising is novel. An increase in s increases the business-stealing effect in (11), making it more attractive to advertise. Moreover, an increase in either type of search costs *ceteris paribus* leads to more market power vis-à-vis consumers that visit a firm and hence higher prices, further increasing the returns to advertising.[8] The effect of t on prices is standard: higher search costs imply more market power. The ambiguous effect of s is surprising. An increase in s does increase market power, but also increases the business-stealing effect of advertising in (11). The latter implies higher advertising levels and hence, via Corollary 1, lower prices. Unsurprisingly, a higher marginal cost of advertising lead to less advertising, which immediately implies higher prices.

4 Welfare

After having derived the market outcome, we now solve for the social optimum and compare that to the market outcome. We thus assume that each firm sets the same advertising level $\tilde{\varphi}$ that is now determined by a social planner. Prices are still set competitively.

Social optimum For simplicity, we will henceforth assume that $c(\varphi) = \frac{1}{2}a\varphi^2$, with $a > 0$ an exogenous parameter. This allows us to find an explicit expression for φ^*. We also make the additional assumption that

$$f(0) \leq \frac{1+2a}{2a}. \tag{13}$$

Note that this condition is very mild. For example, it is always satisfied if $f(0) < 1$. Sufficient for that to hold is that $f' \geq 0$ – a condition that is often imposed in similar models.

From (11), the market outcome now has

$$\varphi^* = \frac{p^*(1 + (1 - F(\hat{\varepsilon}_s))^2)}{2a + p^*}$$

Note that we have an internal solution with $\varphi^* \leq 1$ if and only if $a \geq \frac{1}{2}p^*(1 - F(\hat{\varepsilon}_s))^2$. In the remainder of this paper, we assume that to be the case.

[8] Haan and Moraga-González (2011) find a similar result, but in a somewhat different context.

To derive total welfare, we have to distinguish between two types of informed consumers: those that are informed by both firms, and those that are informed by just one. Denote the (ex ante expected) gross utility they will obtain as U_t and U_s, respectively. Conveniently, U_t thus refers to the utility of consumers that are twice informed, while U_s refers that of consumers that are singly informed: a consumer that is informed by a single firm will obtain on average utility U_s after having gone through the search process while a consumer that is informed by both firms will on average obtain U_t. These utilities equal the stand-alone value v plus the match value that the consumer obtains from her purchase, minus any search costs that she incurs to secure that match value. Total welfare then equals

$$W = \varphi^2 U_t + 2\varphi(1-\varphi)U_s - a\varphi^2, \tag{14}$$

as a share φ^2 are informed by both firms, and a share $2\varphi(1-\varphi)$ by just one of them. Consumers that are informed all buy in equilibrium, hence prices do affect welfare. Maximizing W with respect to φ yields

$$\tilde{\varphi} = \frac{U_s}{a + 2U_s - U_t}, \tag{15}$$

provided that this yields an interior solution. Otherwise $\tilde{\varphi} = 1$.[9]

To further assess $\tilde{\varphi}$, we have to pin down U_s and U_t. Note that search costs for the first firm equal t for all informed consumers. Note that a consumer that faces costs x to search a second firm will do so with probability $F(\hat{\varepsilon}_x)$. Denote the match value that such a consumer will ultimately obtain as ε_x. We can then write

$$U_x = v + E(\varepsilon_x) - t - F(\hat{\varepsilon}_x) \cdot x, \tag{16}$$

for $x \in \{s, t\}$. A consumer that is twice informed has $x = t$. With probability $F(\hat{\varepsilon}_t)^2$, both her match values are lower than $\hat{\varepsilon}_t$; in that case, she gets the highest of the two. Otherwise she ends up with a draw higher than $\hat{\varepsilon}_t$. Hence, her expected match value is

$$\begin{aligned} E(\varepsilon_t) &= F(\hat{\varepsilon}_t)^2 E(\max\{\varepsilon_1, \varepsilon_2\} \,|\, \max\{\varepsilon_1, \varepsilon_2\} < \hat{\varepsilon}_t) + \left(1 - F(\hat{\varepsilon}_t)^2\right) E(\varepsilon | \varepsilon > \hat{\varepsilon}_t) \\ &= 2\int_0^{\hat{\varepsilon}_t} \varepsilon f(\varepsilon) F(\varepsilon) d\varepsilon + (1 + F(\hat{\varepsilon}_t)) \int_{\hat{\varepsilon}_t}^1 \varepsilon f(\varepsilon) d\varepsilon. \end{aligned}$$

[9]We show in the proof of Proposition 3 that the denominator is positive, hence $\tilde{\varphi} > 0$.

From (2) we have

$$t = \int_{\hat{\varepsilon}_t}^{1} \varepsilon f(\varepsilon)\, d\varepsilon - \hat{\varepsilon}_t \left(1 - F(\hat{\varepsilon}_t)\right) \tag{17}$$

This implies, from (16),

$$\begin{aligned} U_t &= v + E(\varepsilon_t) - (1 + F(\varepsilon_t))t \\ &= v + 2\int_0^{\hat{\varepsilon}_t} \varepsilon f(\varepsilon)F(\varepsilon)d\varepsilon + (1 + F(\hat{\varepsilon}_t)) \int_{\hat{\varepsilon}_t}^{1} \varepsilon f(\varepsilon)d\varepsilon - (1 + F(\hat{\varepsilon}_t)) \left[\int_{\hat{\varepsilon}_t}^{1} \varepsilon f(\varepsilon)\, d\varepsilon - \hat{\varepsilon}_t \left(1 - F(\hat{\varepsilon}_t)\right)\right] \\ &= v + 2\int_0^{\hat{\varepsilon}_t} \varepsilon f(\varepsilon)F(\varepsilon)d\varepsilon + \hat{\varepsilon}_t \left(1 - F^2(\hat{\varepsilon}_t)\right). \end{aligned} \tag{18}$$

Along the same lines, for consumers that are only informed by one firm we can derive

$$\begin{aligned} U_s = v + E(\varepsilon_s) - t - F(\varepsilon_s)s &= v + E(\varepsilon_s) - (1 + F(\hat{\varepsilon}_s))\, s + s - t \\ &= v + 2\int_0^{\hat{\varepsilon}_s} \varepsilon f(\varepsilon)F(\varepsilon)d\varepsilon + \hat{\varepsilon}_s \left(1 - F^2(\hat{\varepsilon}_s)\right) + s - t, \end{aligned} \tag{19}$$

Using these expressions, we can now show the following result:

Proposition 3 (Comparative statics social optimum)**.** *The optimal fraction of informed consumers per firm $\tilde{\varphi}$ is decreasing in search costs t and the cost of advertising a. If $\tilde{\varphi} < \frac{1}{2}$, it is increasing in search costs s. If $\tilde{\varphi} > \frac{1}{2}$, it is decreasing in search costs s.*

Hence, contrary to the market outcome, we have that the socially optimal level of advertising is *decreasing* in search costs t. As these increase, the gross utility U_t that each consumer can expect to earn from participating in the market, decreases. Hence, the social returns from investi ng in advertising also decrease. The welfare effects of an increase in s are ambiguous. Note that an increase in φ has an ambiguous effect on the number of singly informed consumers. With φ relatively low, a further increase will mainly switch consumers from being uninformed to being singly informed. But for larger φ, a further increase will mainly switch consumers from being singly to being twice informed.

Comparison with the market outcome We now compare the social optimum with the market outcome. Before we proceed, we first establish the following useful result:

Lemma 1. *Consider the case that the search costs s and t are equal. We then have that φ^* is strictly increasing in those search costs, while $\tilde{\varphi}$ are strictly decreasing in those search costs.*

Suppose we are in the limit case that $s = t$. The market level of advertising then increases in those search costs. This immediately follows from Proposition 2: higher search costs imply more market power and more business stealing, both reasons to increase the level of advertising. But we also find that the social optimal advertising level *de*creases in those search costs. When search costs are higher, the benefits of being informed decrease, hence the social return to informing consumers are also lower.

This result suggests that with $s = t$, the market provides too little advertising when search costs are sufficiently low, and too much advertising when they are sufficiently high. By continuity, it also suggests that at least to some extent, the same is true for $s < t$. It turns out that this is indeed the case. More precisely:

Proposition 4. *Comparing the market outcome with the social optimum, we have the following:*

1. *For sufficiently low search costs s and t, the market provides too little advertising.*

2. *For sufficiently high search costs s and t, the market provides too much advertising.*

3. *Consider the case $s = t$. There is a cut-off value of search costs $\tilde{t}$ such that the market provides too little advertising if $t < \tilde{t}$, and too much if $t > \tilde{t}$.*

Intuitively, with low search costs, the business stealing effect we identified in (11) is also low. The private return of putting out an additional ad then only equals that firm's share of the market expansion effect, i.e. $\frac{1}{2}(1 - \varphi)p^*$, whereas the social returns are $(1 - \varphi)(p^* + CS)$, i.e. the total profits plus consumer surplus generated if the newly informed consumer was not yet informed. This is clearly higher. For very high search costs though, the business stealing effect of advertising will become so strong that the private returns to advertising will ultimately outweigh the social return. Indeed, with φ approaching 1, social benefits of additional advertising will almost disappear, while the business stealing effect would be as strong as ever. There will thus be a lot of wasteful duplication of advertising effort. Hence, from a welfare perspective, the amount of advertising provided by the market will then be too high.

5 Numerical example

Set-up In this section, we use a numerical example to illustrate the effects we derived above. Throughout, we assume a uniform distribution of match values and a quadratic advertising cost function, as in the previous section: $c(\varphi) = a\varphi^2/2$. Again, we will set $v = 1$.

Demand (3) now simplifies to

$$\bar{D}_{xj} = \frac{1}{2}\left(1 - \hat{\varepsilon}_x - \Delta\right)\left(1 + \hat{\varepsilon}_x\right) + \frac{1}{2}\left(\varepsilon_x^2 - \frac{1}{2}\Delta^2\right)$$

while, from (4) we have.

$$\bar{p}_x^* = \frac{1}{1 + \hat{\varepsilon}_x}$$

The equilibrium condition on advertising, (11) simplifies to

$$\varphi^* = \frac{p^*\left(2 - \hat{\varepsilon}_s\left(2 - \hat{\varepsilon}_s\right)\right)}{2\alpha + p^*}.$$

From (2), we have that $\hat{\varepsilon}_x = 1 - \sqrt{2x}$. Hence

$$\varphi^* = \frac{p^*\left(1 + 2s\right)}{2\alpha + p^*}$$

while (12) simplifies to

$$p^* = \frac{1}{2}\frac{1}{1 - \frac{\varphi}{2-\varphi}\sqrt{2t} - \frac{2(1-\varphi)}{2-\varphi}\sqrt{2s}}.$$

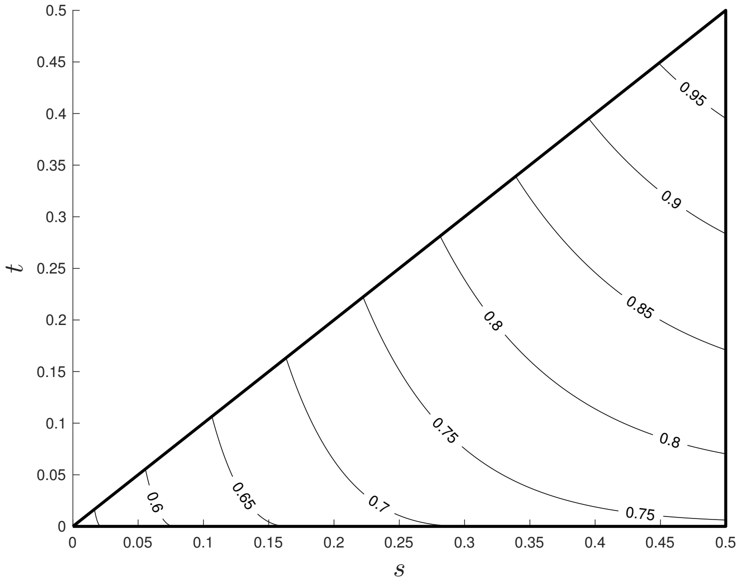
From (18) and (19) we now have

$$\begin{aligned} U_t &= v + \frac{2}{3} + \frac{2}{3}t^{\frac{3}{2}}\sqrt{2} - 2t \\ U_s &= v + \frac{2}{3} + \frac{2}{3}s^{\frac{3}{2}}\sqrt{2} - s - t. \end{aligned}$$

Plugging these into (15) yields the socially optimal level of advertising.

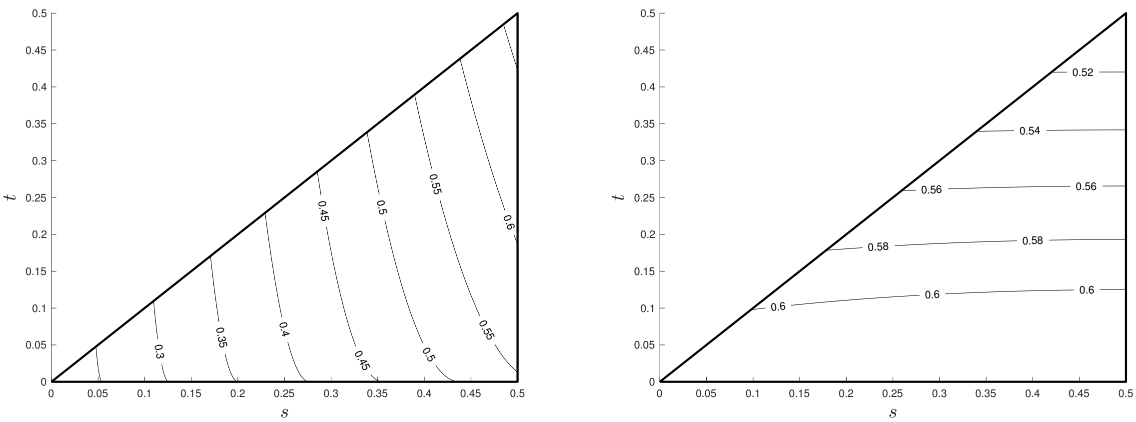
Prices For the case that $a = 1$, Figure 3 shows equilibrium price levels in (s, t)-space, with s on the horizontal, and t on the vertical axis. The curves are iso-price curves. Note that, as $s \geq t$, only the lower right triangle of this graph is relevant. From the Figure we see indeed that prices increase in both types of search costs, s and t. As we assume that the market is covered, profits per firm simply equal $\pi = 0.5p^*$. Hence, the curves in Figure 3 also reflect iso-profit curves.

Figure 3: Prices in the market equilibrium



Advertising For the case that $a = 1$, Figure 4 shows advertising levels in (s, t)-space. The left-hand panel of the figure gives the market equilibrium levels of advertising, while the right-hand panel gives the social optimum. The curves are iso-advertising curves. In the left-hand panel, as we move to the right, equilibrium advertising levels increase. In the right-hand panel, as we move up, socially optimal advertising levels increase.

Figure 4: Advertising levels.



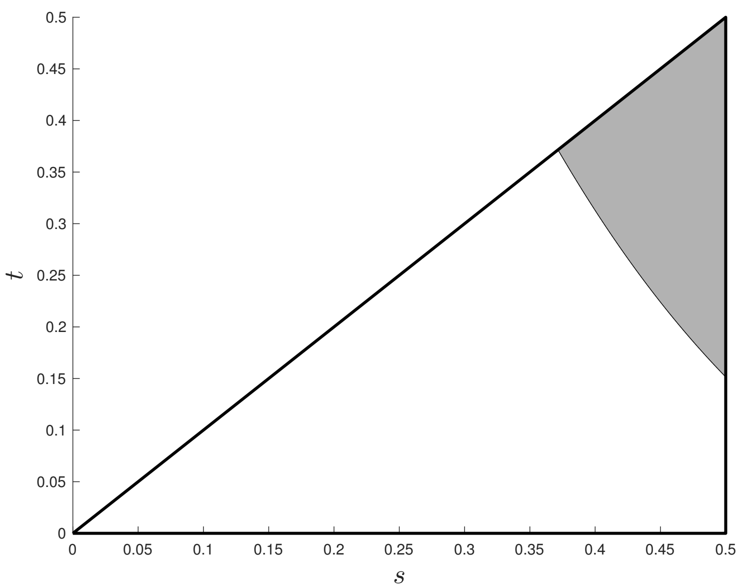
(a) Market equilibrium

(b) Social optimum

What stands out immediately is that market advertising levels are mainly affected by s (the search costs of finding a firm that one has not seen advertised) and are hardly affected by t (the search costs of visiting a firm that one *has* seen advertised), while socially optimal advertising levels are mainly affected by t, but are hardly affected by s. The reason is as follows. The social planner is primarily interested in informing people. With lower t the social returns of doing so are higher as newly informed consumers can then more cheaply acquire a product: they incur search costs t when doing so. On the other hand, the private returns to advertising are mainly affected by whether a consumer that is informed by a firm will also buy there: that market power is higher when s, the costs of finding an alternative firm, are higher.

Over- and underadvertising If we superimpose the two panels of Figure 4 on each other, we can compare the social optimum with the market outcome. Figure 5 gives the (grey) area for which the market equilibrium provides too much advertising. As we established in Proposition 4, this is the case if s and t are high enough. Needless to say, there is underadvertising in the non-shaded part of Figure 5.

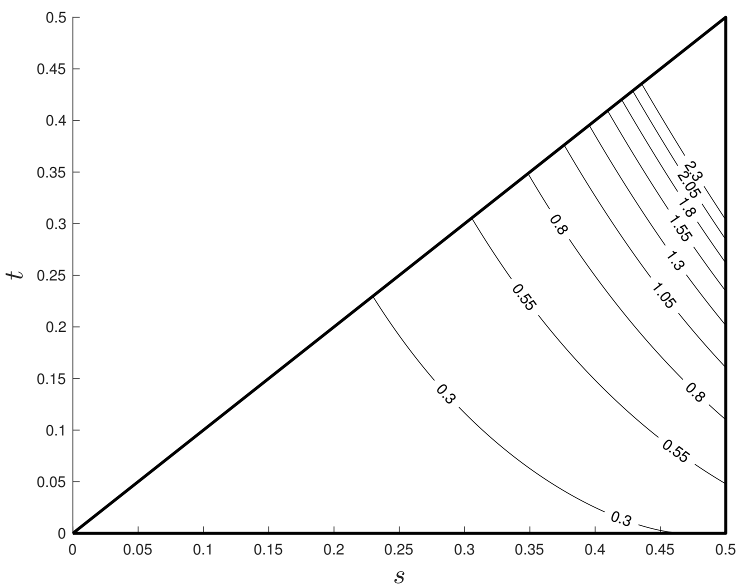
Figure 5: Area with overadvertising



The shape of the area with overadvertising is always similar to that in Figure 5. This can be seen as follows. From Proposition 4, there is always overadvertising on the line $s = t$ for high enough search costs, and underadvertising for search costs low enough. Moreover, whenever there is underadvertising in some point (s, t) the facts that market advertising is increasing in t and that socially optimal advertising is decreasing in t together imply that there is also underadvertising in any point below (s, t). In turn, this implies that there always exists some curve where the market outcome equals the social optimum; that this curve crosses the line $s = t$; that it is downward sloping in (s, t)-space; and that there is overadvertising to the north-east of that curve and under-advertising to the south-west.

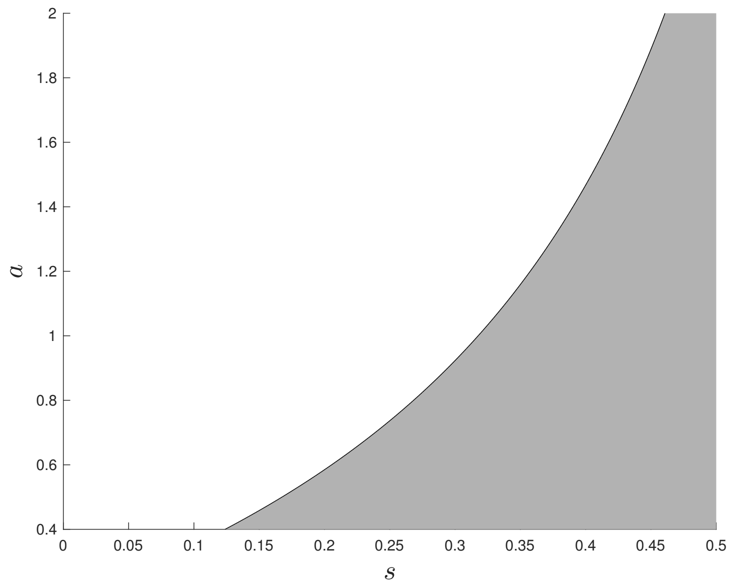
Optimal advertising As noted, on the curve that intersects the two areas in Figure 4, the socially optimal advertising level exactly equals the level provided by the market. We will refer to this as an optimal-advertising curve. Figure 6 gives these optimal-advertising curves for different level of the marginal cost of advertising a. Note that they are consistent with the analysis above. To the northeast of each curve, there is overadvertising for that level of a, while to the southwest, there is underadvertising. From the Figure, as a increases, there is more likely to be underadvertising. Note from the discussion above that overadvertising is especially likely if φ is high as the market expansion effect is then low but the business stealing effect is still prevalent. But a high a implies a lower φ^*, making overadvertising less likely.

Figure 6: Optimal advertising curves



The case $t = 0$ In Figure 7 we consider the case where a consumer can freely visit a firm whose product she has seen advertised. i.e. the extreme case that $t = 0$. The Figure gives the area where there now is overadvertising. Clearly, this is more likely as s increases. This is in line with the observation we made with Figure 4: market levels of advertising are strongly affected by s, while the social optimum is not. Hence, as s increases, market levels of advertising increase while socially optimal levels are hardly affected. Hence, there is more likely to be overadvertising. From the figure, we also see that there is more likely to be underadvertising as a increases. This is in line with Figure 6.

Figure 7: Overadvertising with $t = 0$



6 An advertising cartel

In the analysis above, we saw that in our model the market is likely to provide underadvertising when search costs are high, but overadvertising when these are low. This suggests that it may be welfare improving to allow firms to make agreements on their level of advertising, while still competing in prices. In this section, we study whether and, if so, when such a welfare cartel may indeed be welfare improving. Note that the analysis concerning prices still goes through as in Section 3, allowing us to focus on the cartel's advertising decision.

Denote the gross profits per informed consumer that the cartel secures in the second stage as $\tilde{\pi}$. As the market is fully covered, we simply have $\tilde{\pi} = p^*$. In the first stage, the cartel thus sets φ to maximize total cartel profits

$$\Pi(\varphi) = \varphi(2 - \varphi) \cdot \tilde{\pi}(\varphi) - 2c(\varphi),$$

with $\varphi(2-\varphi)$ the total number of informed consumers. The first-order condition of the cartel's problem is

$$\frac{\partial \Pi}{\partial \varphi} = 2(1-\varphi)p^* + \varphi(2-\varphi)\frac{\partial p^*}{\partial \varphi} - 2c'(\varphi^*) = 0,$$

which yields

$$c'(\varphi) = p^*(1-\varphi) + \frac{1}{2}\varphi(2-\varphi)\cdot\frac{\partial p^*}{\partial \varphi}. \tag{20}$$

It is instructive to compare this to the first-order condition for the market equilibrium, given by (11). First, the cartel will take into account that a change in φ affects p^*, which is the last term. Ignoring that term for now, the marginal benefit for the cartel of increasing one firm's φ is $p^*(1-\varphi)$, with $1-\varphi$ the probability that the newly informed consumer was not yet reached by the other firm. Hence, this is the total market-expansion effect. The marginal benefit of advertising for an individual firm only depends on half of the market-expansion effect: $\frac{1}{2}(1-\varphi)$, but also includes the business-stealing effect $\frac{1}{2}(1-F(\hat{\varepsilon}_s))^2$. Hence, the market equilibrium provides too much advertising from the point of view of the industry whenever the business-stealing effect is bigger than half of the market-expansion effect. This is more likely if $\hat{\varepsilon}_s$ is low, hence if s is high.

We can also show the following:

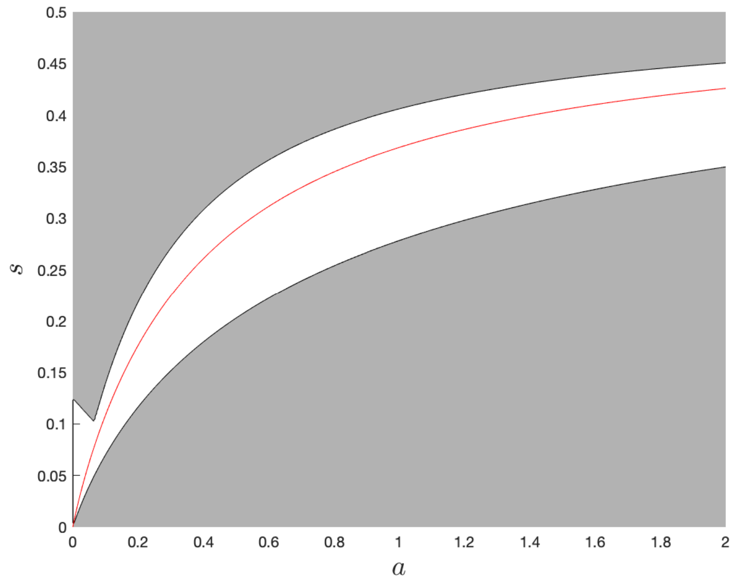
Proposition 5. *A cartel that sets advertising levels always underadvertises.*

We thus have that the cartel always advertises less than the social planner. There are two channels through which this effect operates. First, for given prices, the cartel sets φ to maximize profits, while the social planner maximizes total surplus, which is always higher. Hence it is willing to advertise more. Second, note from Proposition 1 that an increase in φ decreases p^* if $s > t$. Such a decrease in p^* adversely affects the cartel but does not affect social welfare, yielding an additional reason for the cartel to restrict φ relative to the social planner.

Despite the fact that the cartel always underadvertises, allowing for such a cartel in advertising levels may still increase welfare compared to the case in which firms compete. To illustrate this, we again consider the case in which match values are uniformly distributed, $v = 1$, and costs of advertising are given by $c(\varphi) = a\varphi^2/2$. To further simplify the exposition, we also assume that $t = s$ which implies that we rule out the price effect of advertising.

Figure 8 gives the analysis in (a, s)-space. The red line gives the optimal-advertising curve: in this context, i.e. the combinations of a and s for which the market outcome yields the

Figure 8: Welfare-improving cartels.



socially optimal level of advertising. In the grey area, an advertising cartel would increase social welfare.[10] This can be understood as follows. First note that on the optimal-advertising curve, allowing a cartel would not be a good idea: from Proposition 5, advertising would be lower as a result. The same is true for parameter values sufficiently close to that line. As argued above, for high s the cartel advertises less than the market, but the market provides overadvertising. Hence an advertising cartel then brings us closer to the social optimum. For very low s, the market advertises less than a cartel so, again, a cartel would bring us closer to the social optimum.

7 Extension: more firms

A natural question is how our analysis is affected if we allow for more than two firms. In this section we consider that case. Yet, as we will see, this greatly complicates the analysis, both for the case of the market outcome as well as for the social optimum. We do find that with a uniform distribution of match values, we again have that there is underadvertising for sufficiently low, and overadvertising for sufficiently high search costs.

Before continuing the analysis, there is one complication we have to address. Suppose that we again assume that $s > t$. There are n firms. Suppose that a consumer has seen ads from

[10]The discontinuity at the low end of the graph is due to hitting the upper bound $\varphi = 1$.

$m < n$ of those[11]. She will first consider the m firms she has seen advertised, and continue search among these as long as she encounters match values below $\hat{\varepsilon}_t$. Suppose now that this consumer has visited all these m firms, and found an $\varepsilon < \hat{\varepsilon}_t$ at each. She will then continue search at the $n - m$ other firms, *provided the match values encountered so far are also below* $\hat{\varepsilon}_s$. If this is not the case, she will go back to the firm with the highest match value among the first m. Otherwise, she will search among the remaining $n - m$ firms until she finds an $\varepsilon > \hat{\varepsilon}_s$. If she does not find such an ε, she will return to the firm with the highest match value among all n firms. Needless to say, all this will make the analysis prohibitively complicated. In the remainder of this section, We therefore assume for simplicity that $s = t$.

we analyze the n-firm model along the same lines as the 2-firm model. First, we derive the market equilibrium. Then. we analyze the social optimum. Using these results, We then derive when there will be underadvertising, and when there will be overadvertising.

Market equilibrium Given that all other firms charge price p^* and inform φ^* consumers, the profits for firm j of setting price p_j and informing φ_j consumers can be written

$$\pi_j(p_j, \varphi_j; p^*, \varphi^*) = p_j D_j(p_j, \varphi_j; p^*, \varphi^*) - c(\varphi_j)$$

Again defining $\Delta \equiv p_j - p^*$, we can write

$$D_j(p_j, \varphi_j; p^*, \varphi^*) = \sum_{k=1}^{n} \omega_k(\varphi_j, \varphi^*) \left[F(\hat{\varepsilon}_s)^{k-1}(1 - F(\hat{\varepsilon}_s + \Delta)) + \int_{\Delta}^{\hat{\varepsilon}_s + \Delta} F(\varepsilon - \Delta)^{n-1} f(\varepsilon) d\varepsilon \right]. \tag{21}$$

Here, $\omega_k(\varphi_j, \varphi^*)$ is the endogenous probability that firm j is the kth in line to be visited by a consumer (i.e. it will be the kth firm visited by this consumer, provided she does not buy already from any of the first $k-1$ firm she visits). Conditional on being kth in line, the consumer buys immediately from this firm if she turns down the first $k - 1$ firms (which happens with probability $F(\hat{\varepsilon}_s)^{k-1}$) but does accept the offer of this firm (which happens with probability $1 - F(\hat{\varepsilon}_s + \Delta)$). The integral represents the consumers that find a match value below $\hat{\varepsilon}_s$ at all firms but return to firm j as it has the highest match value among all firms.

[11] In our context it may be a bit of a stretch to assume that consumers know the exact number of firms, although they do not know the identity of those firms. Note however that the decision whether or not to continue search does not depend on the number of firms left to search. Hence our analysis still holds when consumers do not know in advance how many firms they can search.

Given this demand function, the equilibrium is again given by (8) and (9). Let's first derive equilibrium prices from (9). Note that

$$\omega_k(\varphi^*, \varphi^*) = \frac{1-(1-\varphi^*)^n}{n},$$

i.e. in equilibrium any position in a consumer's search order is equally likely, but the total size of the market is $1-(1-\varphi^*)^n$ rather than 1, as this is the number of consumers that will be informed in equilibrium. The demand function $D_j(p_i, \varphi^*; p^*, \varphi^*)$ then collapses to that in Anderson and Renault (1999), up to a constant. Hence we get the same equilibrium price[12]:

To determine equilibrium advertising levels, we need to specify the weights $\omega_k(\varphi_i, \varphi^*)$. Consistent with the 2-firm case, we assume that a consumer first visits the firms she has seen advertised and only then may also visit firms she has not seen advertised. Define $\mathbb{P}_\ell$ as the equilibrium probability of a consumer receiving ℓ advertisements from $n-1$ firms:

$$\mathbb{P}_\ell \equiv \binom{n-1}{\ell}(\varphi^*)^\ell(1-\varphi^*)^{n-1-\ell}.$$

We then have

$$\omega_k(\varphi_i, \varphi^*) = \varphi_i \sum_{\ell=k-1}^{n-1} \frac{\mathbb{P}_\ell}{\ell+1} + (1-\varphi_i)\sum_{\ell=1}^{k-1}\frac{\mathbb{P}_\ell}{n-\ell}. \tag{22}$$

This can be understood as follows. The first term reflects a consumer that see j's ad, which happens with probability φ_i. In that case, for firm i to be kth in line, this consumer needs to see at least $k-1$ other ads, hence the limits of the summation. If the consumer is hit by ℓ ads from other firms, the probability that i is kth in line is $1/(\ell+1)$, which yields the first summation. The second term reflects a consumer that doesn't see i's ad. For i to be kth in line, the consumer has to see at most $k-1$ ads from other firms, hence the limits to the summation. If the consumer is hit by ℓ ads, the probability that i is kth in line then is $1/(n-\ell)$, which yields the second summation.

Plugging (22) and (21) into (9) we have, after some tedious manipulations:

Proposition 6. *With n firms and $s = t$, equilibrium advertising levels are given by*

$$c'(\varphi^*) = p_n^* \left[\frac{(1-\varphi^*)^{n-1}}{n} + \frac{1-(1-F(\hat{\varepsilon}_s)^n)\varphi^* - (1-(1-F(\hat{\varepsilon}_s))\varphi^*)^n}{n(1-\varphi^*)\varphi^*}\right] \tag{23}$$

[12]See equation 5 in Anderson and Renault (1999) with $L = \frac{1-(1-\varphi^*)^n}{n}$.

The equilibrium price equals that in a standard model of search with differentiated products where consumers have search costs s:

$$p_n^* = \frac{1}{\frac{1-F(\hat{\varepsilon}_s)^n}{1-F(\hat{\varepsilon}_s)} f(\hat{\varepsilon}_s) - n\int_0^{\hat{\varepsilon}_s} f'(\varepsilon)F(\varepsilon)^{n-1}d\varepsilon}. \tag{24}$$

A number of things are worth observing. First, just as with two firms, equal search costs for all firms implies that equilibrium prices are not affected by advertising levels and simply equal those in the standard model of search with differentiated products. Second, the expression for equilibrium advertising has the same structure as in the 2-firm case. The first term in brackets in equation (23) is the market expansion effect, while the second term is the business stealing effect. Again, the business stealing effect goes to 0 if search costs do (so $F(\hat{\varepsilon}_s)$ goes to 1). Third, equilibrium prices are increasing in search costs and decreasing in the number of firms. However, comparative statics for advertising levels are impossible to sign.[13]

Social optimum Now consider the socially optimal advertising levels. We proceed along the same lines as in the 2-firm case. We again assume quadratic advertising costs: $c(\varphi) = a\varphi^2$. Similar to (14), we can then write total welfare with n firms as

$$W_n = (1-(1-\varphi)^n)U_n - \frac{na}{2}\varphi^2,$$

where we define U_n as the gross utility of a consumer that is informed about the existence of the product, when n firms can be sought. The first-order condition yields[14]

$$(1-\varphi)^{n-1}U_n - a\varphi = 0. \tag{25}$$

Note that being able to search $n+1$ rather than n firms, potentially adds to a consumer's utility if it turns out that the first n firms all provide a match value below $\hat{\varepsilon}_s$. If so, this consumer can now visit yet another firm at search costs s that may provide a higher match value. Hence

$$U_{n+1} = U_n + F(\hat{\varepsilon}_s)^n \left(M_{n+1} - s\right), \tag{26}$$

[13] For instance it is reasonable to expect that advertising levels are decreasing in the number of firms. However counterexamples are easy to find. If s is such that $F(\hat{\varepsilon}_s) = 0.8$ and $a = 0.013$, then $\varphi_3^* > \varphi_2^*$ (numerical calculations are available upon request).

[14] The second-order condition $-n(n-1)(1-\varphi)^{n-2}U_n - na \leq 0$ is obviously satified for any φ.

with M_{n+1} the expected increase in match value of an $n+1$th firm. The distribution of the highest match value at the first n firms is given by:

$$G_n(\varepsilon) = \left(\frac{F(\varepsilon)}{F(\hat{\varepsilon}_s)}\right)^n.$$

Hence we have

$$M_{n+1} = \int_0^{\hat{\varepsilon}_s} \int_x^1 (u-x) f(u) g_n(x) du dx.$$

Using the fact that

$$g_n = \frac{n f(\varepsilon) F(\varepsilon)^{n-1}}{F(\hat{\varepsilon}_s)^n},$$

(26) implies

$$U_{n+1} = U_n - F(\hat{\varepsilon}_s)^n s + n \int_0^{\hat{\varepsilon}_s} \left(\int_x^1 (u-x) f(u) du \right) f(x) F(x)^{n-1} dx. \tag{27}$$

For simplicity, assume that match values are uniformly distributed. We then have[15]

$$U_n = \hat{\varepsilon}_s - \frac{1}{n+1} \hat{\varepsilon}_s^{n+1} \tag{28}$$

so the first-order condition collapses to

$$(1-\varphi)^{n-1} \hat{\varepsilon}_s \left(1 - \frac{1}{n+1} \hat{\varepsilon}_s^n\right) - a\varphi = 0. \tag{29}$$

We can now show the following:

Proposition 7 (Comparative statics social optimum)**.** *With n firms and a uniform distribution of match values, optimal advertising φ_n decreases in search costs s and the cost of advertising a. The effect of an increase in the number of firms is ambiguous.*

[15]From (2), $\hat{\varepsilon}_s$ is then implicitly defined by $s = \frac{1}{2}(1-\hat{\varepsilon}_s)^2$. Using this and evaluating (27), this implies

$$U_{n+1} = U_n + \frac{1}{n+1} \hat{\varepsilon}_s^{n+1} - \frac{1}{n+2} \hat{\varepsilon}_s^{n+2}.$$

Given that $U_1 = \frac{1}{2} - s$, it is easy to verify that this implies

$$U_n = \frac{1}{2} + \frac{1}{2} \hat{\varepsilon}_s^2 - \frac{1}{n+1} \hat{\varepsilon}_s^{n+1} - s.$$

Again using $s = \frac{1}{2}(1-\hat{\varepsilon}_s)^2$, we obtain (28).

The comparative statics effects of a change in search costs are similar to the $s = t$ case with two firms. The effect of an increase in the number of firms may go either way. On the one hand, having more firms increases the returns to advertising, as informed consumers can now find better match values. This is a **better match effect**. On the other hand, having more firms also implies that the same level of per-firm advertising now implies more wasteful duplication of advertising effort. This is an **increased duplication effect**. The latter will dominate if $\tilde{\varphi}$ is already relatively high.

Comparison with the market outcome Also in this case, we can show:

Proposition 8. *With $n > 2$ firms, quadratic advertising costs, $s = t$, and a uniform distribution of match values, we have the following:*

1. *For sufficiently low search costs s, the market provides too little advertising.*
2. *For sufficiently high search costs s, the market provides too much advertising.*

Hence, also with more than 2 firms, and with some additional simplifying assumptions, we have that the market provides underadvertising for sufficiently low, and overadvertising for sufficiently high search costs. The intuition is the same as in the case of two firms. With low search costs, there is little business stealing: the private benefits of an additional ad then fall short of the social benefits. For high search costs, business stealing is high so the private returns to advertising outweigh the social returns.

Note that the result only looks at very low and very high values of search costs. It does not directly imply that there is overadvertising if and only if s is bigger than some $\hat{s}_n$. Yet, numerically we can show that this is indeed the case. However, increasing the number of firms has an ambiguous effect on the cutoff value $\hat{s}_n$. In other words, increasing the number firms may either increase or decrease the range of search costs that leads to overadvertising.

This is illustrated in Figure 9, where we plot $\hat{s}_3$ and $\hat{s}_4$ as a function of the advertising parameter a. From the figure, for relatively low values of a, increasing the number of firms from 3 to 4 increases the range of search costs for which there is underadvertising. However, for relatively high values of a doing so increases the range of search costs for which there is overadvertising.

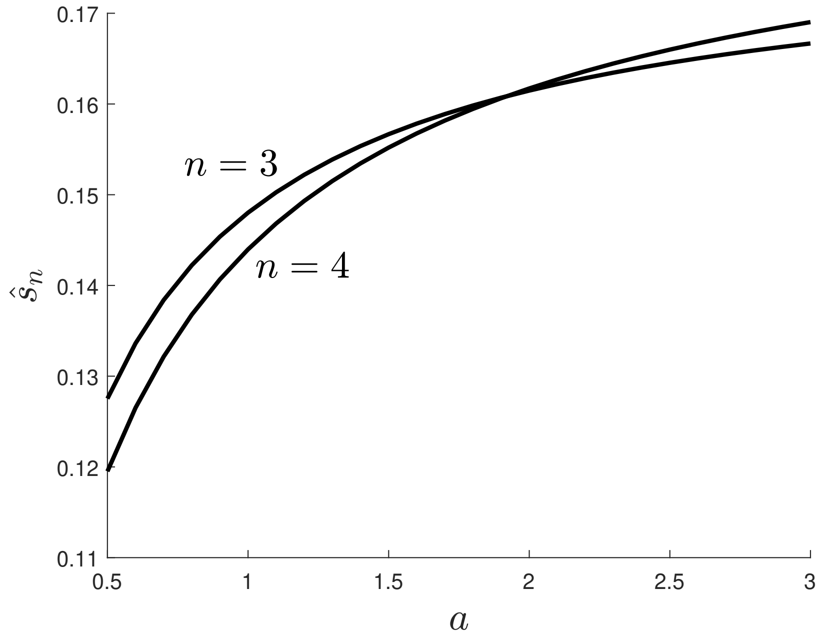


Figure 9: Increasing the number of firms has an ambiguous effect on the extent of under- and overadvertising.

8 Conclusion

In this paper, we studied a search market where two firms inform consumers of the existence of a market by means of advertising. A consumer that observes the ad of a firm learns about the existence of the market — which may also trigger her to seek out the other firm. Doing so however implies search costs that are higher than for a firm she has seen advertised.

In our set-up, advertising has a market-expansion as well as a business-stealing effect. As the search costs of finding an as-of-yet unknown firm increase, the business-stealing effect becomes stronger. For high search, the market then provides too much advertising from a welfare perspective. If search costs are low, there is too much free riding in advertising, and the market provides too little advertising from a welfare perspective. An advertising cartel provides too little advertising from a welfare perspective. It advertises less than the market if search costs are high (to eliminate the business-stealing effect) but more if search costs are low (as it then solves the free-riding problem). This implies that an advertising cartel may increase welfare, provided that the market outcome is sufficiently far removed from the social optimum.

We also study the effects of an increase in the number of firms. In that case, however, we have to assume that search costs are equal for all firms, regardless of whether a consumer has seen them advertised. But also then we find that, with a uniform distribution of match values, the market provides underadvertising for low, and overadvertising for high search costs. The effect of increasing the number of firms is ambiguous.

While our analysis has focused on the scenario where consumers are initially unaware of the product being sold, we believe the insights from our work have much broader applicability. In any case where seeing the ads of one firm may induce consumers to search for alternatives, the same mechanism is likely to be at play. With search costs sufficiently low, advertising primarily expands the market, leading firms to advertise too little out of fear for free-riding. With search costs sufficiently high, advertising primarily yields business-stealing, causing firms to advertise too much, resulting in a wasteful duplication of advertising efforts.

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Appendix: Proofs

Proof of Proposition 2

We first consider the effects of s and t. It is easier to do the analysis in terms of the reciprocals of prices rather than the prices themselves. Write $r \equiv 1/p^*$, $r_t \equiv 1/\bar{p}_t$ and $r_s \equiv 1/\bar{p}_s$. Also define $\alpha(\varphi) \equiv \varphi / (2 - \varphi)$. The expression for equilibrium prices in Proposition 1 then collapses to

$$r = \alpha(\varphi)\, r_t + (1 - \alpha(\varphi))\, r_s. \tag{30}$$

Note that r_t and r_s are decreasing in t and s, respectively, and $\alpha'(\varphi) > 0$. For the equilibrium condition on φ, we can write, from (11)

$$c(\varphi) \cdot r = \frac{1}{2}(1 - \varphi) + \frac{1}{2}(1 - F(\hat{\varepsilon}_s))^2. \tag{31}$$

We first study the effects of a change in t. Take the total differential of (30) and (31):

$$dr = \alpha'(\varphi)(r_t - r_s)d\varphi + \alpha(\varphi)r_t' dt$$
$$c'(\varphi)dq + c''(\varphi)d\varphi = -\frac{1}{2}d\varphi$$

Gathering terms this yields:

$$\begin{bmatrix} 1 & \alpha'(\varphi)(r_s - r_t) \\ c'(\varphi) & c''(\varphi) + \frac{1}{2} \end{bmatrix} \begin{pmatrix} \frac{dr}{dt} \\ \frac{d\varphi}{dt} \end{pmatrix} = \begin{pmatrix} \alpha(\varphi)r_t' \\ 0 \end{pmatrix}.$$

Using Cramer's rule, this implies

$$\frac{dr}{dt} = \frac{\alpha(\varphi)r_t' \cdot \left(c''(\varphi) + \frac{1}{2}\right)}{c''(\varphi) + \frac{1}{2} - \alpha'(\varphi)(r_s - r_t)c(\varphi)} < 0.$$

With $r_t' < 0$ and $c''(\varphi) > 0$, the numerator is negative. As $\alpha' > 0$ and $p_s^* > p_t^*$ implies $r_s < r_t$, the denominator is positive. Taken together, his implies the stated result which in turn implies that the equilibrium price is increasing in t.

Moreover we have

$$\frac{d\varphi}{dt} = \frac{-\overbrace{a(\varphi)r_t'}^{<0} \cdot \overbrace{c'(\varphi)}^{>0}}{\underbrace{c''(\varphi) + \frac{1}{2}}_{>0} - \underbrace{\alpha'(\varphi)(r_s - r_t)c(\varphi)}_{<0}} > 0.$$

This establishes the comparative statics results with respect to t.

Next we study how a change in s affects the equilibrium values of r and φ. Take the total differential of (30) and (31):

$$dr = \alpha'(\varphi)(r_t - r_s)d\varphi + (1 - \alpha(\varphi))r_s' ds \tag{32}$$

$$c'(\varphi)dr + c''(\varphi)d\varphi = -\frac{1}{2}d\varphi - (1 - F(\hat{\varepsilon}_s))f(\hat{\varepsilon}_s)\frac{d\hat{\varepsilon}_s}{ds}dr. \tag{33}$$

From (2) we have $\int_{\hat{\varepsilon}_s}^{1} (\varepsilon - \hat{\varepsilon}_s) f(\varepsilon)d\varepsilon - s = 0$. Taking the derivative with respect to s yields $-(1 - F(\hat{\varepsilon}_s)) \frac{d\hat{\varepsilon}_s}{ds} = 1$. Plugging this into (33) we can rewrite the system (32)-(33) as

$$\begin{bmatrix} 1 & \alpha'(\varphi)(r_s - r_t) \\ c'(\varphi) & c''(\varphi) + \frac{1}{2} \end{bmatrix} \begin{pmatrix} \frac{dr}{ds} \\ \frac{d\varphi}{ds} \end{pmatrix} = \begin{pmatrix} (1 - \alpha(\varphi))r_s' \\ f(\hat{\varepsilon}_s) \end{pmatrix}.$$

Using Cramer's rule, this yields

$$\frac{dr}{ds} = \frac{\overbrace{(1 - \alpha(\varphi))\, r_s'}^{<0} \cdot \overbrace{\left(c''(\varphi) + \frac{1}{2}\right)}^{>0} - \overbrace{\alpha'(\varphi)(r_s - r_t)f(\hat{\varepsilon}_s)}^{<0}}{\underbrace{c''(\varphi) + \frac{1}{2} - \alpha'(\varphi)(r_s - r_t)c(\varphi)}_{>0}},$$

rendering the net effect ambiguous. If $t \to s$, then $r_t - r_s \to 0$ so $dr/ds < 0$ and prices increase in s. But if $\varphi \to 1$, $1 - \alpha(\varphi) \to 0$ so $dr/ds > 0$. In this scenario prices decrease in s.

Finally, we have

$$\frac{d\varphi}{ds} = \frac{\overbrace{f(\hat{\varepsilon}_s)}^{>0} - \overbrace{(1 - \alpha(\varphi))r_s' c'(\varphi)}^{<0}}{\underbrace{c''(\varphi) + \frac{1}{2} - \alpha'(\varphi)(r_s - r_t)c(\varphi)}_{>0}} > 0.$$

So an increase in s also leads to an increase in advertising.

For the effects of the marginal costs of advertising, note that φ^* is only determined by (11), where the left-hand side gives the marginal cost, and the right-hand side the marginal benefit of advertising. For ease of exposition, denote these as $MC(\alpha, \varphi)$ and $MB(\varphi)$ respectively: α is some exogenous variable that shifts marginal costs, $\partial MC/\partial\alpha > 0$. Due to concavity, marginal costs are increasing in φ: $dMC/d\varphi > 0$. From the right-hand side of (11), marginal benefits are decreasing in φ: it is clear that the bracketed term decreases in φ, while from Proposition 1, p^* is nonincreasing in φ. Hence $dMB/d\varphi < 0$. Equilibrium requires $MC(\alpha, \varphi) - MB(\varphi) = 0$. Using the implicit function theorem,

$$\frac{d\varphi}{d\alpha} = -\frac{\frac{dMC}{d\alpha}}{\frac{dMC}{d\varphi} - \frac{dMB}{d\varphi}} < 0.$$

Hence, φ decreases with an increase in marginal costs: this immediately implies an increase in prices. ■

Proof of Proposition 3

We first prove the following lemma:

Lemma 2. *We necessarily have* $U_s\left(1 + F\left(\hat{\varepsilon}_t\right)\right) > U_t$.

Proof. First note that the lowest possible value of U_s is achieved searching beyond the first firm is prohibitively high and $\hat{\varepsilon}_s = 0$. Hence, using (19),

$$\begin{aligned} U_s &\geq v + \int_0^1 \varepsilon f(\varepsilon) d\varepsilon - t \\ &= v + \int_0^{\hat{\varepsilon}_t} \varepsilon f(\varepsilon) d\varepsilon + \hat{\varepsilon}_t (1 - F(\hat{\varepsilon}_t)), \end{aligned}$$

again using (17). Using (18), we can then write

$$\begin{aligned} &(1 + F(\hat{\varepsilon}_t))\, U_s - U_t \geq \\ &F(\hat{\varepsilon}_t)v + (1 + F(\hat{\varepsilon}_t)) \int_0^{\hat{\varepsilon}_t} \varepsilon f(\varepsilon) d\varepsilon + (1 + F(\hat{\varepsilon}_t))\, \hat{\varepsilon}_t (1 - F(\hat{\varepsilon}_t)) - 2 \int_0^{\hat{\varepsilon}_t} \varepsilon f(\varepsilon) F(\varepsilon) d\varepsilon - \hat{\varepsilon}_t (1 - F(\hat{\varepsilon}_t)^2) \\ =&F(\hat{\varepsilon}_t)v + \int_0^{\hat{\varepsilon}_t} (1 + F(\hat{\varepsilon}_t) - 2F(\varepsilon)) \varepsilon f(\varepsilon) d\varepsilon + (1 + F(\hat{\varepsilon}_t))\, \hat{\varepsilon}_t (1 - F(\hat{\varepsilon}_t)) - \hat{\varepsilon}_t (1 - F(\hat{\varepsilon}_t)^2) \\ =&F(\hat{\varepsilon}_t)v + \int_0^{\hat{\varepsilon}_t} (1 + F(\hat{\varepsilon}_t) - 2F(\varepsilon)) \varepsilon f(\varepsilon) d\varepsilon > 0, \end{aligned}$$

where the last inequality follows from the fact that $\varepsilon \leq \hat{\varepsilon}_t \leq 1$ for all relevant ε. This establishes the result. ■

Using this Lemma, we can now prove the Proposition. For the effect of a, differentiating (15) with respect to a yields:

$$\frac{\partial \tilde{\varphi}}{\partial a} = -\frac{U_s}{(a + 2U_s - U_t)^2} < 0.$$

For the effects of t and s, we proceed as follows. First note from (2) that

$$\frac{d\hat{\varepsilon}_x}{dx} = \frac{1}{-\int_{\hat{\varepsilon}_x}^{1} f(\varepsilon)d\varepsilon} = \frac{-1}{1 - F(\hat{\varepsilon}_x)}.$$

Using (18), we then have

$$\frac{\partial U_t}{\partial t} = \frac{\partial U_t}{\partial \hat{\varepsilon}_t} \cdot \frac{d\hat{\varepsilon}_t}{dt} = \left(1 - F^2(\hat{\varepsilon}_t)\right) \cdot \frac{-1}{1 - F(\hat{\varepsilon}_t)} = -\left(1 + F(\hat{\varepsilon}_t)\right).$$

Similarly, from (19),

$$\frac{\partial U_s}{\partial s} = \frac{\partial U_s}{\partial \hat{\varepsilon}_s} \cdot \frac{d\hat{\varepsilon}_s}{ds} - 1 = -F(\hat{\varepsilon}_s).$$

Differentiating (15) with respect to s and t then yields, also using (15),

$$\begin{aligned} \frac{d\tilde{\varphi}}{ds} &= \frac{\partial \tilde{\varphi}}{\partial U_s} \cdot \frac{\partial U_s}{\partial s} \\ &= \frac{-(a - U_t) F(\hat{\varepsilon}_s)}{(a + 2U_s - U_t)^2} \end{aligned} \tag{34}$$

$$\begin{aligned} \frac{d\tilde{\varphi}}{dt} &= \frac{\partial \tilde{\varphi}}{\partial U_s} \cdot \frac{\partial U_s}{\partial t} + \frac{\partial \tilde{\varphi}}{\partial U_t} \cdot \frac{\partial U_t}{\partial t} \\ &= \frac{-(a - U_t)}{(a + 2U_s - U_t)^2} + \frac{-U_s \cdot (1 + F(\hat{\varepsilon}_t))}{(a + 2U_s - U_t)^2} \\ &= -\frac{a + U_s(1 + F(\hat{\varepsilon}_t)) - U_t}{(a + 2U_s - U_t)^2} < 0, \end{aligned} \tag{35}$$

where the last inequality follows from Lemma 2.

For the effect of s, note that with $a > U_t$, we have from (34) that $d\tilde{\varphi}/ds < 0$ and, from (15) that $\tilde{\varphi} > 1/2$. However, if $a < U_t$, we have from (34) that $d\tilde{\varphi}/ds > 0$ and, from (15) that $\tilde{\varphi} < 1/2$. Taken together, this implies the result of s. ■

Proof of Lemma 1

First note from (10) and (11) that with $s = t$, the equilibrium is given by

$$\begin{aligned} p^* &= \bar{p}_t^* \\ c'(\varphi^*) &= p^* \left[\frac{1}{2}(1-\varphi) + \frac{1}{2}(1-F(\hat{\varepsilon}_t))^2 \right] \end{aligned}$$

Making the dependency of p^* and φ^* on $\hat{\varepsilon}_t$ explicit, this simplifies to:

$$c'(\varphi(\hat{\varepsilon}_t)) = p(\hat{\varepsilon}_t) \left[\frac{1}{2}(1-F(\hat{\varepsilon}_t))^2 + \frac{1}{2}(1-\varphi(\hat{\varepsilon}_t)) \right]$$

Differentiate both sides with respect to $\hat{\varepsilon}_t$ and solve for $\varphi'(\hat{\varepsilon}_t)$:

$$\varphi'(\hat{\varepsilon}_t) = \frac{p'(\hat{\varepsilon}_t)\left[\frac{1}{2}(1-F(\hat{\varepsilon}_t))^2 + \frac{1}{2}(1-\varphi(\hat{\varepsilon}_t))\right] - p(\hat{\varepsilon}_t) f(\hat{\varepsilon}_t)(1-F(\hat{\varepsilon}_t))}{c''(\varphi(\hat{\varepsilon}_t)) + \frac{1}{2}p(\hat{\varepsilon}_t)}.$$

With $p'(\hat{\varepsilon}_t) < 0$ and $c''(\cdot) > 0$, we have that $\varphi'(\hat{\varepsilon}_t) < 0$: the fraction of informed consumers is decreasing in $\hat{\varepsilon}_t$, which implies it is increasing in t. To show that $\tilde{\varphi}$ is decreasing in t along the line $s = t$, note that in that case, from (15) that $\tilde{\varphi} = U_t/(a + U_t)$. This expression is increasing in U_t; with U_t decreasing in t, this implies that $\tilde{\varphi}$ is decreasing in t. This establishes the result.
■

Proof of Proposition 4

To prove the first statement of the Proposition, it is sufficient to show that it holds for $s = t = 0$; to prove the second statement it is sufficient to show that it holds for $\hat{\varepsilon}_s = \hat{\varepsilon}_t = 0$, which is consistent with having the highest feasible search costs. Once we have established that, the third statement immediately follows from Lemma 1.

First note that with $s = t$ we have overadvertising, hence $\varphi^* > \tilde{\varphi}$, whenever

$$\frac{1 + (1-F(\hat{\varepsilon}_t))^2}{2a + p^*} \cdot p^* > \frac{U_t}{a + U_t}$$

Rewriting yields[16]

$$U_t < \frac{ap^*\left(1 + (1-F(\hat{\varepsilon}_t))^2\right)}{2a - (1-F(\hat{\varepsilon}_t))^2 p^*} \tag{36}$$

[16] Note that the denominator is strictly positive under the assumption that $\varphi^* < 1$.

Suppose $s = t \to 0$, hence $\hat{\varepsilon}_t \to 1$. We then have

$$E(\varepsilon_t) = E\left(\varepsilon_{(2)}\right),$$

with $\varepsilon_{(2)}$ the highest-order statistic of two draws from F. Condition (36) then collapses to

$$v + E\left(\varepsilon_{(2)}\right) < \frac{p^*}{2}. \tag{37}$$

With $s = t$ and $\hat{\varepsilon}_t \to 1$, we have $p^* = 1/2 \int_0^1 f^2(\varepsilon)d\varepsilon$, Necessarily $\int_0^1 (f(\varepsilon) - 1)^2 \, d\varepsilon \geq 0$, which implies $\int_0^1 \left(f(\varepsilon)^2 - 2f(\varepsilon) + 1\right) d\varepsilon \geq 0$. But $\int_0^1 f(\varepsilon)\, d\varepsilon = 1$, hence $\int_0^1 f(\varepsilon)^2 \, d\varepsilon \geq 1$ so $p^* \leq 1/2$. With $v = 1$, this implies (37) is never satisfied. Hence, there is underadvertising in that case.

Now suppose $\hat{\varepsilon}_t \to 0$, which implies $t \to E(\varepsilon)$. In that case $E(\varepsilon_t) = E(\varepsilon | \varepsilon > 0) = E(\varepsilon)$, so $U_t = v + E(\varepsilon) - t = v$. Condition (36) then collapses to

$$v < \frac{2ap^*}{2a - p^*}.$$

With $v = 1$, this implies that we need $p^* > \frac{2a}{2a+1}$. From (4), when then have $p^* = 1/f(0)$, which implies that we need (13), which we assume is satisfied. ■

Proof of Proposition 5

For total welfare, we can write, using (14),

$$W = \varphi^2 (U_t - p^*) + 2\varphi(1 - \varphi)(U_s - p^*) + \Pi(\varphi).$$

where $U_t - p^*$ and $U_s - p^*$ are the net consumer surpluses (including search costs) of consumers that are informed by 2 firms and 1 firm respectively. Denote the φ that maximizes the cartel's profits, and hence solves (20), as φ_K^*. The derivative of total welfare with respect to φ is given by

$$\frac{\partial W}{\partial \varphi} = 2\varphi(U_t - p^*) + 2(1 - 2\varphi)(U_s - p^*) - 2\varphi(1 - \varphi) \cdot \frac{\partial p^*}{\partial \varphi} + \frac{\partial \Pi}{\partial \varphi}.$$

Evaluating this in φ_K yields

$$\frac{\partial W}{\partial \varphi} = 2(U_s - p^*) + 2\varphi(U_t - U_s) - 2\varphi(1 - \varphi) \cdot \frac{\partial p^*}{\partial \varphi} > 0,$$

as $U_t > U_s$ and $\partial p^*/\partial\varphi < 0$. But concavity of W then implies that $\tilde{\varphi} > \varphi_K$, which establishes the result. ■

Proof of Proposition 6

Equilibrium advertising requires, from (8) and using (21),[17]

$$c'(\varphi^*) = p_n^* \sum_{k=1}^{n} \left[F(\hat{\varepsilon}_s)^{k-1}(1 - F(\hat{\varepsilon}_s)) + \frac{1}{n} F(\hat{\varepsilon}_s)^n \right] \cdot \frac{\partial\omega_k}{\partial\varphi_i}(\varphi^*, \varphi^*). \tag{38}$$

Note that the total market size equals

$$\sum_{k=1}^{n} \omega_k(\varphi_i, \varphi^*) = 1 - (1 - \varphi_i)(1 - \varphi^*)^{n-1}.$$

But that implies

$$\sum_{k=1}^{n} \frac{\partial\omega_k}{\partial\varphi_i} = (1 - \varphi^*)^{n-1}.$$

so from (38) we have, using (22),

$$c'(\varphi^*) = p_n^* \frac{F(\hat{\varepsilon}_s)^n (1 - \varphi^*)^{n-1}}{n} + p_n^* \sum_{k=1}^{n} F(\hat{\varepsilon}_s)^{k-1}(1 - F(\hat{\varepsilon}_s)) \left(\sum_{\ell=k-1}^{n-1} \frac{\mathbb{P}_\ell}{\ell+1} - \sum_{\ell=1}^{k-1} \frac{\mathbb{P}_\ell}{n-\ell} \right). \tag{39}$$

Note that[18]

$$\sum_{k=1}^{n} F(\hat{\varepsilon}_s)^{k-1}(1 - F(\hat{\varepsilon}_s)) \sum_{\ell=k-1}^{n-1} \frac{\mathbb{P}_\ell}{\ell+1} = \sum_{\ell=0}^{n-1} \frac{1 - F(\hat{\varepsilon}_s)^{\ell+1}}{\ell+1} \cdot \mathbb{P}_\ell,$$

[17] In deriving this expression, we use $\int_0^{\hat{\varepsilon}_s} F(\varepsilon)^{n-1} f(\varepsilon) d\varepsilon = \frac{1}{n} F(\hat{\varepsilon}_s)^n$.

[18] Here, we use the fact that

$$\sum_{k=1}^{n} a_k \sum_{\ell=k-1}^{n-1} b_\ell = \sum_{\ell=0}^{n-1} b_\ell \sum_{k=1}^{\ell+1} a_k,$$

see (2.32) on pg. 36 of Graham et al. (1994). Hence the left-hand side can be rewritten as:

$$\sum_{\ell=0}^{n-1} \frac{\mathbb{P}_\ell}{\ell+1} \left(\sum_{k=1}^{\ell+1} F(\hat{\varepsilon}_s)^{k-1}(1 - F(\hat{\varepsilon}_s)) \right) = \sum_{\ell=0}^{n-1} \frac{1 - F(\hat{\varepsilon}_s)^{\ell+1}}{\ell+1} \cdot \mathbb{P}_\ell$$

Along the same lines we have[19]

$$\sum_{k=1}^{n} F(\hat{\varepsilon}_s)^{k-1}(1-F(\hat{\varepsilon}_s))\sum_{\ell=1}^{k-1}\frac{\mathbb{P}_\ell}{n-\ell} = \sum_{\ell=1}^{n-1}\frac{F(\hat{\varepsilon}_s)^\ell - F(\hat{\varepsilon}_s)^n}{n-\ell}\mathbb{P}_\ell$$
$$= \quad -\frac{1-F(\hat{\varepsilon}_s)^n}{n}(1-\varphi^*)^{n-1} + \sum_{\ell=0}^{n-1}\frac{F(\hat{\varepsilon}_s)^\ell - F(\hat{\varepsilon}_s)^n}{n-\ell}\mathbb{P}_\ell$$

Plugging these two expressions in (39) we get:

$$c'(\varphi^*) = p_n^*\left[\frac{(1-\varphi^*)^{n-1}}{n} + \sum_{\ell=0}^{n-1}\left(\frac{1-F(\hat{\varepsilon}_s)^{\ell+1}}{\ell+1} - \frac{F(\hat{\varepsilon}_s)^\ell - F(\hat{\varepsilon}_s)^n}{n-\ell}\right)\mathbb{P}_\ell\right] \tag{40}$$

To further simplify, we write the summation as

$$\sum_{\ell=0}^{n-1}\left(\frac{1-F(\hat{\varepsilon}_s)^{\ell+1}}{\ell+1} - \frac{F(\hat{\varepsilon}_s)^\ell - F(\hat{\varepsilon}_s)^n}{n-\ell}\right)\mathbb{P}_\ell = A - F(\hat{\varepsilon}_s)B - C + F(\hat{\varepsilon}_s)^n D, \tag{41}$$

with

$$A \equiv \sum_{\ell=0}^{n-1}\frac{\mathbb{P}_\ell}{\ell+1};\ B \equiv \sum_{\ell=0}^{n-1}\frac{F(\hat{\varepsilon}_s)^\ell\mathbb{P}_\ell}{\ell+1};\ C \equiv \sum_{\ell=0}^{n-1}\frac{F(\hat{\varepsilon}_s)^\ell\mathbb{P}_\ell}{n-\ell};\ D \equiv \sum_{\ell=0}^{n-1}\frac{\mathbb{P}_\ell}{n-\ell}.$$

Note that

$$\frac{1}{\ell+1} = \int_0^1 x^\ell dx$$

hence we can write

$$A = \sum_{\ell=0}^{n-1}\left(\int_0^1 x^\ell dx\right)\mathbb{P}_\ell = \int_0^1\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}(\varphi x)^\ell(1-\varphi)^{n-1-\ell}dx$$
$$= \int_0^1(\varphi x + (1-\varphi))^{n-1}dx = \frac{1-(1-\varphi)^n}{\varphi n}.$$

where we use the Binomial Theorem in the third equality. Along the same lines

$$B = \sum_{\ell=0}^{n-1}\left(\int_0^1 x^\ell dx\right)F(\hat{\varepsilon}_s)^\ell\mathbb{P}_\ell = \int_0^1\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}(\varphi x F(\hat{\varepsilon}_s))^\ell(1-\varphi)^{n-1-\ell}dx$$
$$= \int_0^1(\varphi F(\hat{\varepsilon}_s)x + (1-\varphi))^{n-1}dx = \frac{(\varphi F(\hat{\varepsilon}_s) + (1-\varphi))^n - (1-\varphi)^n}{\varphi F(\hat{\varepsilon}_s)n}.$$

[19]Here, we use the fact that $\sum_{k=1}^n a_k \sum_{\ell=1}^{k-1} b_\ell = \sum_{\ell=1}^{n-1} b_\ell \sum_{k=\ell+1}^n a_k$ so we can write the left-hand side as

$$\sum_{\ell=1}^{n-1}\frac{\mathbb{P}_\ell}{n-\ell}\left(\sum_{k=\ell+1}^{n} F(\hat{\varepsilon}_s)^{k-1}(1-F(\hat{\varepsilon}_s))\right)$$

Next, note that

$$\frac{1}{n-\ell} = \int_0^1 x^{n-1-\ell} dx,$$

so

$$\begin{aligned} D &= \sum_{\ell=0}^{n-1} \left(\int_0^1 x^{n-1-\ell} dx \right) \mathbb{P}_\ell = \int_0^1 \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (\varphi)^\ell ((1-\varphi)x)^{n-1-\ell} dx \\ &= \int_0^1 (\varphi + (1-\varphi)x)^{n-1} dx = \frac{1-\varphi^n}{(1-\varphi)n}, \end{aligned}$$

again using the Binomial Theorem. Along the same lines

$$\begin{aligned} C &= \sum_{\ell=0}^{n-1} \left(\int_0^1 x^{n-1-\ell} dx \right) F(\hat{\varepsilon}_s)^\ell \mathbb{P}_\ell = \int_0^1 \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (\varphi F(\hat{\varepsilon}_s))^\ell ((1-\varphi)x)^{n-1-\ell} dx \\ &= \int_0^1 (F(\hat{\varepsilon}_s)\varphi + (1-\varphi)x)^{n-1} dx = \frac{(F(\hat{\varepsilon}_s)\varphi + (1-\varphi))^n - (F(\hat{\varepsilon}_s)\varphi)^n}{(1-\varphi)n} \end{aligned}$$

We can thus write the terms in (41) as

$$\begin{aligned} A &= \frac{(1-\varphi) - (1-\varphi)^{n+1}}{n(1-\varphi)\varphi} \\ -F(\hat{\varepsilon}_s)B &= \frac{-(1-\varphi)(\varphi F(\hat{\varepsilon}_s) + (1-\varphi))^n + (1-\varphi)^{n+1}}{n(1-\varphi)\varphi} \\ -C &= \frac{-\varphi(F(\hat{\varepsilon}_s)\varphi + (1-\varphi))^n + \varphi(F(\hat{\varepsilon}_s)\varphi)^n}{n(1-\varphi)\varphi} \\ F(\hat{\varepsilon}_s)^n D &= \frac{F(\hat{\varepsilon}_s)^n (1-\varphi^n)\varphi}{n(1-\varphi)\varphi} \end{aligned}$$

The sum of these four terms collapses to

$$\frac{(1-\varphi) - (\varphi F(\hat{\varepsilon}_s) + (1-\varphi))^n + \varphi F(\hat{\varepsilon}_s)^n}{n(1-\varphi)\varphi}.$$

Rearranging terms and plugging this back into (40) we obtain (23).

We already derived the equilibrium price in the main text. ■

Proof of Proposition 7

Let $\tilde{\varphi}_n$ denote the welfare-maximizing level of advertising given by (25) and (28). It is easy to check that $\tilde{\varphi}_n$ is unique and $\tilde{\varphi}_n \in (0,1)$. We now have

$$\frac{d\tilde{\varphi}}{da} = \frac{-\tilde{\varphi}}{(n-1)(1-\tilde{\varphi})^{n-2}U_n + a} < 0$$

and

$$\frac{d\tilde{\varphi}}{ds} = \frac{(1-\varphi)^{n-1}\frac{\partial U_n}{\partial \hat{\varepsilon}_s}\frac{\partial \hat{\varepsilon}_s}{\partial s}}{(n-1)(1-\tilde{\varphi})^{n-2}U_n + a} < 0$$

The most straightforward way to illustrate that comparative statics with respect to n are ambiguous, is to compare the equilibrium for the case that $n = 2$ with the case $n = 3$. From (29), we have

$$\begin{aligned} \tilde{\varphi}_2 &= \frac{3\hat{\varepsilon}_s - \hat{\varepsilon}_s^3}{3a + 3\hat{\varepsilon}_s - \hat{\varepsilon}_s^3}; \\ \tilde{\varphi}_3 &= 1 - \frac{2\sqrt{a^2 - a\hat{\varepsilon}_s^4 + 4a\hat{\varepsilon}_s} - 2a}{4\hat{\varepsilon}_s - \hat{\varepsilon}_s^4}. \end{aligned}$$

Suppose for example that $\hat{\varepsilon}_s = 0.8$. In that case, we have $\tilde{\varphi}_2 > \tilde{\varphi}_3$ if and only if $a < 5.8017$.■

Proof of Proposition 8

First note that, from (24), equilibrium prices with a uniform distribution can be shown to equal

$$p_n^* = \frac{1-\hat{\varepsilon}_s}{1-\hat{\varepsilon}_s^n}.$$

Let's first consider the case of low search costs. As $s \to 0$, we have that $\hat{\varepsilon}_s \to 1$ and $p_n^* \to 1/n$. The condition for a market equilibrium (23) hence collapses to

$$a\varphi^* = \frac{1}{n}\left[\frac{(1-\varphi^*)^{n-1}}{n}\right], \tag{42}$$

while the condition for the social optimum (29) equals

$$a\tilde{\varphi} = \frac{n}{n+1}(1-\tilde{\varphi})^{n-1}. \tag{43}$$

As the right-hand side of (43) is clearly larger for any φ than the right-hand side of (42), and moreover the left-hand sides are increasing and the right-hand sides are decresing in φ, this implies that $\tilde{\varphi} > \varphi^*$, so there is underadvertising.

Now consider the case of high search costs. The highest feasible search costs are such that $\hat{\varepsilon}_s \to 0$, so $p_n \to 1$. The condition for the market equilibrium (23) then collapses to

$$a\varphi^* = \left[\frac{1}{n\varphi^*}\left(1 - (1 - \varphi^*)^n\right)\right],$$

while the condition for the social optimum (29) yields $a\tilde{\varphi} = 0$, which clearly implies $\tilde{\varphi} = 0$. This is obvious: as any consumer is now indifferent between searching or not, there is no use in spending money to inform them. As clearly $\varphi^* > 0$, this implies $\varphi^* > \tilde{\varphi}$, so there is overadvertising. ■